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======================MATHEMATICAL MODELING ======================

Applying Laplace Transformation on Epidemiological Models as Caputo Derivatives

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Abstract. This paper delves into the application of fractional calculus, with a focus on Caputo derivatives, in epidemiological models using ordinary differential equations. It highlights the critical role Caputo derivatives play in modeling intricate systems with memory effects and assesses various epidemiological models, including SIR variants, demonstrating how Caputo derivatives capture fractional-order dynamics and memory phenomena found in real epidemics. The study showcases the utility of Laplace transformations for analyzing systems described by ordinary differential equations (ODEs) with Caputo derivatives. This approach facilitates both analytical and numerical methods for system analysis and parameter estimation. Additionally, the paper introduces a tabular representation for epidemiological models, enabling a visual and analytical exploration of variable relationships and dynamics. This matrix-based framework permits the application of linear algebra techniques to assess stability and equilibrium points, yielding valuable insights into long-term behavior and control strategies. In summary, this research underscores the significance of Caputo derivatives, Laplace transformations, and matrix representation in epidemiological modeling. We assume that by using this type of methodology we can get analytic solutions by hand when considering a function as constant in certain cases and it will not be necessary to search for numerical methods.

Key words: *Caputo derivatives, Laplace transformation, epidemiology, ODEs, fractional calculus, matrices.*

1. INTRODUCTION

1.1. Historical Background

The origin of the fractional calculus can be traced to the question of the extension of the concept. A well-known example is the extension of the concept of real numbers to complex numbers, and another is the extension of the concept of integer factorials to complex factorials. The question of extending the meaning to generalized integration and differentiation is the following question: Can the meaning of integral order derivatives $\frac{d^n y}{dx^n}$ be extended to make sense if n is any number—integer, fractional or complex; The above notation was created by Leibnitz [1].

Perhaps it was a naive symbol game that made L'Hospital ask Leibnitz about the idea that n is a fraction. "What if n is 0.5?" asked L'Hospital. In 1695, Leibnitz said: "This will lead to a paradox". However, he added prophetically, "From this apparent paradox, useful consequences will one day be drawn." In 1697, Leibnitz used the symbol d^2y to imply the infinite product of Wallis for the $\frac{\pi}{2}$ and argued that differential calculus could be used to reach the same conclusion. The first reference to a derivative of arbitrary order is found in an 1819 manuscript. The French mathematician S. F. Lacroix, wrote a 700-page treatise on differential and integral calculus in

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which he devoted just two pages to the subject of fractional derivatives [2].

Euler and Fourier mentioned arbitrary derivatives but did not provide applications or illustrations. It was Niels Henrik Abel who made the first application in 1823 [3]. Abel used fractional calculus to solve an integral equation that arose from the formulation of the question of equal times. This question, also known as the problem of equal time, involves determining the shape of a frictionless wire lying in a vertical plane such that a bead placed on the wire slides to the lowest point of the wire in the same time, regardless of where the bead is placed. The short-time problem concerns the shortest slip time.

1.2. Definitions of fractional derivatives

Definition 1. *The Caputo fractional derivative of a function $f(x)$ with order α, when $0 < \alpha < 1$, is defined as:*

$$^{C}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt$$

Which can be expressed in the following form:

$$^{C}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\int_{a}^{x}\frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha+1}\right)$$

Where n is the smallest integer greater than or equal to α, $f^{(n)}(t)$ denotes the n-th derivative of $f(t)$, $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

Definition 2. *The Riemann-Liouville fractional derivative of a function $f(x)$ with order α, when $0 < \alpha < 1$, is defined as:*

$$^{RL}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{a}^{x}\frac{f(t)}{(x-t)^{\alpha+1-n}}dt,$$

Where n is the smallest integer greater than or equal to α, $f^{(n)}(t)$ denotes the nth derivative of $f(t)$, $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

Definition 3. *The Grünwald-Letnikov fractional derivative of a function $f(x)$ with order α, when $0 < \alpha < 1$, is defined as:*

$$^{GL}D_{x}^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh)$$

Where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ is the binomial expansion.

Definition 4. *The Caputo-Fabrizio fractional derivative of a function $f(t) \in C^1[(a,b)]$ with order α, when $0 < \alpha < 1$, is defined as:*

$$^{CF}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}f^{(n)}(t)E_{\alpha,n}(x-t)^{n-\alpha-1}dt,$$

Where n is the smallest integer greater than or equal to α, $f^{(n)}(t)$ denotes the nth derivative of $f(t)$, $\Gamma(\cdot)$ is the gamma function, $E_{\alpha,n}(x)$ is the Mittag-Leffler function and α is a positive real constant.

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Definition 5. *The Riesz fractional derivative of a function $f(t)$ defined in the interval (a,b) with $\alpha > 0$, with order α is defined as:*

$$^{R}D_x^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dt}\right) \int_a^t \frac{f'(\tau)}{(t-\tau)^{1-\alpha}} d\tau$$

Where $\Gamma(\cdot)$ is the gamma function, $f'(\tau)$ denotes the first derivative of $f(\tau)$ on τ and α is a positive real constant.

1.3. Definitions of fractional integrals

Definition 6. *The Riemann-Liouville fractional integral with order $\alpha > 0$ for a function $f(t)$ set in the interval (a,b) is defined as:*

$$^{RL}I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

Where $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

Definition 7. *The Caputo fractional integral with order $\alpha > 0$ for a function $f(t)$ set in the interval (a,b) is defined as:*

$$^{C}I^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$$

Where $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

Definition 8. *The Grünwald-Letnikov fractional integral with order $\alpha > 0$ for a function $f(t)$ set in the interval (a,b) is defined as:*

$$^{GL}I^{\alpha} f(t) = \lim_{h \to 0^+} h^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(t-kh)$$

Where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ is the binomial expansion, k positive integer and α is a positive real constant.

Definition 9. *The Marchaud fractional integral with order $\alpha > 0$ for a function $f(t)$ set in the interval (a,b) defined as:*

$$^{M}I^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau$$

Where $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

Definition 10. *The Hadamard fractional integral with order $\alpha > 0$ for a function $f(t)$ set in the interval (a,b) defined as:*

$$^{H}I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau$$

Where $\Gamma(\cdot)$ is the gamma function and α is a positive real constant.

2. CAPUTO FRACTIONAL DERIVATIVES AND EPIDEMIOLOGICAL MODELS

2.1. Historical background

The Caputo derivative is a mathematical concept that finds its roots in fractional calculus, a branch of mathematics that deals with generalizations of ordinary derivatives and integrals to non-integer orders. The derivative Caputo, named after the Italian mathematician Michele Caputo, is one of the widely used fractional derivatives because of its favorable properties and its applications in various fields. Michele Caputo, born in Italy in 1927, is a distinguished mathematician who has made significant contributions to the field of fractional calculus. He was educated at the University of Ferrara, where he obtained a degree in mathematics in 1950, a degree in Physics from the University of Bologna in 1955 and a degree in free teaching (libera docenza) in Geodesy from the Italian Ministry of Education in 1959. Caputo's interest in fractional calculus arose during his studies and research on the theory of elasticity. In 1967, Caputo published a fundamental paper [4]. This work introduced what is now known as the fractional derivative Caputo, which generalizes the notion of differentiation to fractional classes.

Caputo himself was also involved in other sciences such as seismology, geology and geophysics. The work of Michele Caputo in fractional calculus, in particular the development of the derivative Caputo, had a major impact on the field and paved the way for further developments in the theory of fractional calculus and its applications. His contribution has been recognized and appreciated by the scientific community and he continues to be considered a leading figure in the field of fractional calculus. Today, the derivative Caputo remains a key tool in the study of fractional calculus and continues to contribute to the understanding of complex phenomena in various scientific disciplines. The historical background and pioneering work of Michele Caputo played a key role in shaping the field of fractional calculus and its applications, leaving a lasting legacy for future generations of mathematicians and scientists.

2.2. Why we used Caputo derivatives. Real life applications

The Caputo derivative has found wide applications in various scientific and engineering disciplines, such as physics, engineering, signal processing and finance. Its ability to describe the behaviour of complex systems with memory and non-locality has made it a valuable tool for modelling and analysis of dynamic processes. Very important similar research has been conducted on applying Caputo derivatives in epidemiological models and finding the asymptotic solutions of the ODEs (see [5]).

The use of Caputo derivatives can help to transform a system of equations and conditions into a simpler system in order to determine the behaviour of a function (convergence, divergence, stability). Such work has been done on the topics of magnetic materials and their behaviour and thermodynamic models (see [6]), epidemiology and ODE models (ordinary differential equations) (see [7]), the transmission of Covid-19(see [8], [9]) and solutions of FDE (functional differential equations) systems with real-life applications(see [10]).

2.3. Epidemiological models of ordinary differential equations

There are three main types of deterministic models for infectious diseases that are spread by direct person-to-person contact in a population. Here these simpler models are formulated as initial value problems for systems of ordinary differential equations and analysed mathematically. Theorems on the asymptotic stability regions for the equilibrium points are formulated and phase plane portraits of the solution paths are presented. Parameters are estimated for various diseases and used to compare the levels of inoculation required to make the herd immune to these diseases. Although the three models presented are simple and their mathematical analyses are elementary, these models provide notation, concepts, intuition and a foundation for considering more sophisticated models. Some potential improvements involve disease-related factors such as infectious agent, mode of transmission, latency, infectious period, susceptibility and resistance, but also social, cultural, ecology providing a sound intuitive understanding and complete evidence for the three most basic epidemiological models for microparasitic infections. The study of disease occurrence is called epidemiology. An epidemic is an unusually large, short-term outbreak of a disease. A disease is called endemic if it persists in a population. The spread of an infectious disease includes not only disease-related factors such as infectious agent, mode of transmission, latency, infectious period, susceptibility and resistance, but also social, cultural, demographic, economic and geographical factors. The three models considered here are the simplest prototypes of three different types of epidemiological models [11].

Compartmental models are a very general modelling technique. They are often applied to the mathematical modelling of infectious diseases. The population is divided into compartments with labels - for example, S, I or R, (Susceptible, Infected and Removed), where S is the number of susceptible individuals, i.e. when a susceptible and an infectious individual come into ’infectious contact’, the susceptible individual is infected with the disease and is moved to the infectious compartment, I is the number of infectious individuals, i.e. the number of infected individuals capable of infecting susceptible individuals, and R for the number of distant (and immunised) or deceased individuals, i.e. the number of infected individuals who have either recovered from the disease and entered the distant compartment or died (also called ’recovered’ or ’resistant’ in the international literature). People can move between apartments. The order of the labels usually indicates the flow patterns between compartments; for example, SEIS means susceptible, exposed, infectious, and then susceptible again.

The beginnings of these models can be traced back to the early 20th century, with important work by Ross [12], [13], Ross and Hudson [14], Kermack [15] and Kendall [16]. The Reed-Frost model was also an important and widely misunderstood ancestor of modern epidemiological modeling approaches [17].

Models are most often run with ordinary differential equations (which are deterministic), but they can also be used in a stochastic (random) framework, which is more realistic but much more complex to analyze. Models try to predict things like the way a disease spreads or the total number of infected people or the duration of an epidemic and estimate various epidemiological parameters such as the reproductive number. Such models can show how different public health interventions can affect the outcome of an epidemic, e.g. what is the most effective technique for issuing a limited number of vaccines to a given population. Researchers have applied the Laplace-Adomian decomposition method in Caputo-Fabrizio fractional derivatives for childhood diseases that follow the same mathematical procedure that is done in this article (see [18]). Our contribution is that this procedure can be expressed using matrices and this could pave the way to an analytic solution.

The first differential equation model we will report is the SIR. The SIR model is one of the simplest compartmental models and many models are derivatives of this basic form. The model consists of three compartments. This model is quite predictive for infectious diseases that are transmitted from human to human and where recovery confers durable resistance, such as measles, mumps and rubella. These variables (S, I and R) represent the number of people in each compartment at a given time. To represent that the number of susceptible, infectious and remote individuals may change over time (even if the total population size remains constant), we make the exact numbers a function of t (time): $S(t)$, $I(t)$ and $R(t)$. For a given disease in a given population, these functions can be processed to predict potential epidemics and bring them under control.

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As implied by the variable function of t, the model is dynamic in that the numbers in each compartment can fluctuate over time. The importance of this dynamic aspect is most obvious in an endemic disease with a short infectious period, such as measles in the UK before the introduction of a vaccine in 1968. Such diseases tend to occur in epidemic cycles due to the variation in the number of susceptible individuals ($S(t)$) over time. During an outbreak, the number of susceptible individuals decreases rapidly as more of them become infected and thus enter the infected and remote compartment. Each member of the population typically evolves from susceptible to infectious to recovered. This can be represented as a flowchart in which the boxes represent the different compartments and the arrows represent the transition between compartments.

The system of differential equations expressing the SIR model was first introduced by William Ogilvy Kermack and Anderson Gray McKendrick in 1927 [15] and is written as follows.

$$\begin{cases} \frac{dS}{dt} &= -\frac{\beta IS}{N}, \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I, \\ \frac{dR}{dt} &= \gamma I \end{cases} \tag{1}$$

Where $S(t)+I(t)+R(t) = N$, when N the total stable population and β, γ fixed for the degree of infection and recovery respectively. From the form of the system in the form of the equation.

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

A more specific form of the SIR model for vital dynamics and stable population is more commonly used to interpret modern infectious diseases. The resulting system is

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \mu S - \frac{\beta IS}{N}, \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ \frac{dR}{dt} &= \gamma I - \mu R \end{cases} \tag{2}$$

Where the constant Λ is the birth rate and μ the death rate. The basic reproduction rate is given by the relationship $R_0 = \frac{\beta}{\mu+\gamma}$. Also, the DFE (disease-free equilibrium) of the system for $R_0 \leq 1$ is

$$(S(t), I(t), R(t)) = (\frac{\Lambda}{\mu}, 0, 0)$$

The next model we will report is the so-called SEIR, E those who have been exposed to the virus (exposed). For many major infections, there is a significant latency period during which individuals are infected but not yet infectious (E). Similar work, but on Atangana-Baleanu fractional derivatives has been done for the SEIR model for Covid-19, which shows that fractional calculus might even describe new upcoming diseases and pandemics [19]. The resulting SEIR system is as follows.

$$\begin{cases} \frac{dS}{dt} &= \mu N - \mu S - \frac{\beta IS}{N}, \\ \frac{dE}{dt} &= \frac{\beta IS}{N} - (\mu + \alpha)E \\ \frac{dI}{dt} &= \alpha E - (\gamma + \mu)I, \\ \frac{dR}{dt} &= \gamma I - \mu R \end{cases} \tag{3}$$

Where the delay period is a random variable with an exponential distribution with parameter α (i.e. the average delay period is α^{-1}), and also with a birth rate Λ equal to the rate of death $N\mu$

(so that the total number N of the population is constant). Under these conditions we have the equation

$$S + E + I + R = N$$

with a base playback rate of $R_0 = \frac{\alpha}{\mu+\alpha}\frac{\beta}{\mu+\gamma}$ and the equilibrium point (DFE) of the system for $R_0 \leq 1$ is

$$(S(t), I(t), E(t), R(t)) = (N, 0, 0, 0)$$

The next model is a variant of the SEIR model, where there is no immunity. In this model an infection leaves no immunity, so recovered individuals return to the susceptible environment, moving back into the $S(t)$ compartment. The following differential equations, with the same parameters as before and the constant ϵ, describe this model with the following system

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \mu S - \frac{\beta IS}{N} + \gamma I, \\ \frac{dE}{dt} &= \frac{\beta IS}{N} - (\epsilon + \mu)E \\ \frac{dI}{dt} &= \epsilon E - (\gamma + \mu)I, \end{cases} \tag{4}$$

The next model we will refer to is called MSIR. For many infections, including measles, babies are not born in the susceptible part but are immune to the disease in the first few months of life due to protection by the mother's antibodies (passed through the placenta and additionally through colostrum). This is called passive immunity. This additional detail can be illustrated by including a class M (for maternal immunity) at the beginning of the classical SIR model. Consequently, the partition $M(t)$ is added to the strictly mathematical system, as well as the parameter δ.

$$\begin{cases} \frac{dM}{dt} &= \Lambda - \delta M - \mu M, \\ \frac{dS}{dt} &= \delta M - \frac{\beta IS}{N} - \mu S, \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ \frac{dR}{dt} &= \gamma I - \mu R \end{cases} \tag{5}$$

The next model is called MSEIR and is a combination of the SEIR model and the passive immunity M. The parameters used in the ODE system are the same as in the models mentioned above.

$$\begin{cases} \frac{dM}{dt} &= \Lambda - \delta M - \mu M, \\ \frac{dS}{dt} &= \delta M - \frac{\beta IS}{N} - \mu S, \\ \frac{dE}{dt} &= \frac{\beta IS}{N} - (\epsilon + \mu)E \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ \frac{dR}{dt} &= \gamma I - \mu R \end{cases} \tag{6}$$

The last model we will mention is the Double SI model. This model applies to diseases transmitted from insect to human, but also from insects to insects (usually mosquitoes). For this we have two types of compartments S,I, but for two different classes (mosquitoes and humans), namely S_h, I_h for humans and S_m, I_m for mosquitoes. The system consists of four SDEs and the parameters change for the two categories. The parameters Λ_h, $\beta 1$, β_2, k_1, Λ_m, μ and k_2 represent the birth rate for susceptible humans, the transmission rate, the death rate, the birth rate for mosquitoes, the mosquito-to-human transmission rate and the mosquito death rate, respectively. An example of this model which we will quote is for Zika virus transmission (see [7]). Depending on the virus, appropriate changes are made to the variables, but the general model is obvious.

$$\begin{cases} \frac{dS_h}{dt} &= \Lambda_h - \beta_1 S_h I_h - \beta_2 S_h I_m - k_1 S_h, \\ \frac{dI_h}{dt} &= \beta_1 S_h I_h + \beta_2 S_h I_m - k_1 I_h, \\ \frac{dS_m}{dt} &= \Lambda_m - \mu S_m I_h - k_2 S_m, \\ \frac{dI_m}{dt} &= \mu S_m I_h - k_2 I_m \end{cases} \tag{7}$$

As it is expected, each virus, depending on its nature, can be approximated by one or more epidemiological models (there are, however, cases where none of them gives direct results, in contrast to the application of such models to analytical equations aiming at finding numerical solutions and behaviours of viruses [20]. Using these ODE systems, researchers can gain insights into transmission rates, disease severity and the impact of different control strategies, ultimately helping to develop effective public health interventions for different types of viruses.

2.4. Epidemiological model transformation with fractional derivatives

In this section we will see how some of the models we saw in the section 2.3 in fractional derivatives Caputo. With the help of the definition 1 of the fractional derivatives Caputo the following transformations were obtained. The SIR model with the system (2) is transformed as follows.

$$\begin{cases} {}^C D_t^\alpha S(t) &= \Lambda - \mu S - \frac{\beta IS}{N}, \\ {}^C D_t^\alpha I(t) &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ {}^C D_t^\alpha R(t) &= \gamma I(t) - \mu R(t) \end{cases} \tag{8}$$

In this representation, ${}^C D_t^\alpha$ denotes the Caputo fractional derivative with respect to time t. The parameters Λ, μ, β and γ represent the birth rate, the natural death rate, the transmission rate and the recovery rate, respectively (as in all the models we have seen). The integration condition as to τ takes into account the memory effect in the Caputo fractional derivatives, capturing the effect of previous values of $I(\tau)$ and $S(\tau)$ in the current state of the system.

We then transform the SEIR model and the (3) in Caputo derivatives. The parameters are known from above. The system is written as follows.

$$\begin{cases} {}^C D_t^\alpha S(t) &= \mu N - \mu S - \frac{\beta IS}{N}, \\ {}^C D_t^\alpha E(t) &= \frac{\beta IS}{N} - (\mu + \alpha) E, \\ {}^C D_t^\alpha I(t) &= \alpha E - (\gamma + \mu) I, \\ {}^C D_t^\alpha R(t) &= \gamma I(t) - \mu R(t) \end{cases} \tag{9}$$

The SEI model, a simpler variant of the SEIR model, (i.e., the system (4)) is transformed as follows.

$$\begin{cases} {}^C D_t^\alpha S(t) &= \Lambda - \mu S - \frac{\beta IS}{N} + \gamma I, \\ {}^C D_t^\alpha E(t) &= \frac{\beta IS}{N} - (\epsilon + \mu) E, \\ {}^C D_t^\alpha I(t) &= \epsilon E(t) - (\gamma + \mu) I(t) \end{cases} \tag{10}$$

The transformation of a variant of the classical SIR model with added passive immunity compartment (model MSIR), i.e. the system(5) is written as follows.

$$\begin{cases} {}^C D_t^\alpha M(t) &= \Lambda - \delta M(t) - \mu M(t), \\ {}^C D_t^\alpha S(t) &= \delta M - \frac{\beta IS}{N} - \mu S, \\ {}^C D_t^\alpha I(t) &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ {}^C D_t^\alpha R(t) &= \gamma I(t) - \mu R(t) \end{cases} \tag{11}$$

Similarly to before, we will transform the model MSEIR, i.e. (6) (with the same parameters) and the following will result.

$$\begin{cases} {}^C D_t^{\alpha} M(t) & = \Lambda - \delta M(t) - \mu M(t), \\ {}^C D_t^{\alpha} S(t) & = \delta M - \frac{\beta IS}{N} - \mu S, \\ {}^C D_t^{\alpha} E(t) & = \frac{\beta IS}{N} - (\epsilon + \mu) E \\ {}^C D_t^{\alpha} I(t) & = \frac{\beta IS}{N} - \gamma I - \mu I, \\ {}^C D_t^{\alpha} R(t) & = \gamma I(t) - \mu R(t) \end{cases} \tag{12}$$

The last model we will transform is the Double SI, which in this case is the system we have chosen (system (7)). This model will be of use in the following paragraphs, and possibly in some future research. Therefore, the Caputo fractional derivative system will be done as follows.

$$\begin{cases} {}^C D_t^{\alpha} S_h(t) & = \Lambda_h - \beta_1 S_h(t) I_h(t) - \beta_2 S_h(t) I_m(t) - k_1 S_h(t), \\ {}^C D_t^{\alpha} I_h(t) & = \beta_1 S_h(t) I_h(t) + \beta_2 S_h(t) I_m(t) - k_1 I_h(t), \\ {}^C D_t^{\alpha} S_m(t) & = \Lambda_m - \mu S_m(t) I_h(t) - k_2 S_m(t), \\ {}^C D_t^{\alpha} I_m(t) & = \mu S_m(t) I_h(t) - k_2 I_m(t) \end{cases} \tag{13}$$

3. LAPLACE TRANSFORMATION AND CAPUTO DERIVATIVES

3.1. Definition of the Laplace transformation

The definition of the classical Laplace transformation for appropriate functions is given by the following rigorous definition.

Definition 11. *Suppose it is either a continuous function f (or more realistically) a continuous function by parts. The transformation is done as follows.*

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

Notice that this integral, defined for t from 0 to infinity, becomes a new equation of $F(s)$, i.e., our algebraic equation and $s = \sigma i + \omega$ is a complex frequency parameter, with $\sigma, \omega \in \mathbb{R}$.

When the term ”the Laplace transform” is used without qualification, it usually means the one-sided or one-sided transformation. The Laplace transformation can alternatively be defined as the two-sided Laplace transformation or two-sided Laplace transform, extending the limits of integration to the entire real axis. If this is done, the common unilateral transformation becomes just a special case of the bilateral transformation, where the definition of the transformed function is multiplied by the Heaviside step function.

There is also a definition for the bilateral Laplace transformation where the integral is defined over the whole interval $(-\infty, +\infty)$ and is written as follows.

Definition 12. *Suppose it is either a continuous function f (or more realistically) a continuous function by parts. The bipartite transformation is done as follows.*

$$\mathcal{L}[f(t)] = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt = F(s)$$

Notice that this integral, defined for t from 0 to infinity, becomes a new equation of $F(s)$, i.e., our algebraic equation and $s = \sigma i + \omega$ is a complex frequency parameter, with $\sigma, \omega \in \mathbb{R}$.

Another notation for the bilateral Laplace transformation is $\mathcal{B}(f)$ instead of F.

Two integrable functions have the same Laplace transformation only if they differ in a set with Lebesgue measure zero. This means that, in the domain of the transformation, there is an inverse transformation. In fact, except for integrable functions, the Laplace transformation is a one-to-one mapping from one functional space to another and to many other functional spaces, although there is usually no easy characterization of the range.

Typical function spaces where this applies include the spaces of blocked continuous functions, the space of $L^{\infty}(0,\infty)$, or more generally the moderate distributions in $(0,\infty)$. The Laplace transformation is also definite and injective for suitable spaces of modest distributions. In these cases, the image of the Laplace transformation lives in a space of analytic functions in the convergence region.The inverse Laplace transformation is given by the following complex integral, which is known by various names (the Bromwich integral, the Fourier integral-Mellin and the inverse formula Mellin). To date, the inverse Laplace transformation is the most difficult process to understand when solving differential equations with this method [21].

Definition 13. *The inverse Laplace transformation of a function $F(s)$ is expressed as follows.*

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} e^{st}F(s)\,ds$$

where c is a constant for which all singularities of $F(s)$ fall on the left-hand side of the line $\Re(s) = c$.

The Laplace transformation is one of the most useful methods for solving differential equations [22], as discussed below for fractional derivatives. The following definition by [23], [7] of Caputo derivatives is explored. An extended form of the Laplace transform, directly related to the fractional derivatives, is also presented, knowing that there is an interaction of these two [24].

Definition 14. *The Laplace transformation of Caputo fractional derivatives of order α is given by the relation*

$$\mathcal{L}[{}^{C}D_t^{\alpha}f(t)](s) = s^{\alpha}\mathcal{L}f(t) - \sum_{i=1}^{n-1} s^{\alpha-i-1}f^{(i)}(0)$$

where $n-1 < \alpha \le n \in \mathbb{N}$. Then, the relationship can be developed in the form of

$$\mathcal{L}[{}^{C}D_t^{\alpha}f(t)] = \frac{s^{n}\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-1}f'(0) - \cdots - f^{(n-1)}}{s^{n-\alpha}}$$

The above definition and expansion are derived from the well-known Laplace transformation for the n-order derivative, with $n \in \mathbb{Z}$. Except that in the case where the order of the Caputo derivative is $\alpha \in (0,1)$ the computation of the Laplace transformation remains an open question, which has not been extensively addressed by the scientific community, unlike the Laplace transformation of integer-order derivatives.

3.2. Transformation of the model

In this section, we will study the Laplace transforms of the aforementioned systems. In all systems the function $f^{n}(0)n = 1,2,...$ is zero. So the formula of Laplace transformation for the systems we have is as follows.

$$\mathcal{L}[^C D_t^\alpha f_i(t)] = \frac{s^n \mathcal{L}[f_i(t)] - s^{n-1} f_i(0)}{s^{n-\alpha}} \tag{14}$$

That is, for $n = 1$ the formula becomes

$$\mathcal{L}[^C D_t^\alpha f_i(t)] = \frac{s\mathcal{L}[f_i(t)] - f_i(0)}{s^{1-\alpha}} = s^\alpha \mathcal{L}[f_i(t)] - s^{\alpha-1} f_i(0) \tag{15}$$

Where we will solve based on our definition of $\mathcal{L}$ for the known functions we have, derived from the Laplace transformation matrix. Also, the notation () is used for the Laplace transformation of known functions $(S(t), I(t), R(t), M(t), S_h(t), S_m(t), I_h(t), I_m(t))$. More detailed for our models are transformed below.

First, we apply the definition 15 for the system (8), of the SIR model and the following system is obtained, respectively for the given equations.

$$\begin{cases} s^\alpha \hat{S}(s) - s^{\alpha-1} S(0) &= \frac{\Lambda}{s} - \mu \hat{S}(s) - \frac{\beta}{N} \mathcal{SI} \\ s^\alpha \hat{I}(s) - s^{\alpha-1} I(0) &= \frac{\beta}{N} \mathcal{SI} - (\gamma + \mu) \hat{I}(s) \\ s^\alpha \hat{R}(s) - s^{\alpha-1} R(0) &= \gamma \hat{I}(s) - \mu \hat{R}(s) \end{cases} \tag{16}$$

Where, for the sake of uniformity of the complex notation of the equations obtained after the transformation, we have denoted the following integral. When $z \in \mathbb{C}$.

$$\mathcal{SI} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{Re(z)-iT}^{Re(z)+iT} \hat{S}(z) \hat{I}(s-z) dz \tag{17}$$

We then find by applying the definition 15 for the system (9), for the SEIR model and the following system is obtained, respectively for the given equations.

$$\begin{cases} s^\alpha \hat{S}(s) - s^{\alpha-1} S(0) &= \frac{\mu N}{s} - \mu \hat{S}(s) + \frac{\beta}{N} \mathcal{SI} \\ s^\alpha \hat{E}(s) - s^{\alpha-1} E(0) &= \frac{\beta}{N} \mathcal{SI} - (\mu + \alpha) \hat{E}(s) \\ s^\alpha \hat{I}(s) - s^{\alpha-1} I(0) &= \alpha \hat{E}(s) - (\gamma + \mu) \hat{I}(s) \\ s^\alpha \hat{R}(s) - s^{\alpha-1} R(0) &= \gamma \hat{I}(s) - \mu \hat{R}(s) \end{cases} \tag{18}$$

We then find by applying the definition 15 for the system (10), for the SEI model and the following system is obtained, respectively for the given equations.

$$\begin{cases} s^\alpha \hat{S}(s) - s^{\alpha-1} S(0) &= \frac{\Lambda}{s} - \mu \hat{S}(s) - \frac{\beta}{N} \mathcal{SI} + \gamma \hat{I}(s) \\ s^\alpha \hat{E}(s) - s^{\alpha-1} E(0) &= \frac{\beta}{N} \mathcal{SI} - (\epsilon + \mu) \hat{E}(s) \\ s^\alpha \hat{I}(s) - s^{\alpha-1} I(0) &= \epsilon \hat{E}(s) - (\gamma + \mu) \hat{I}(s) \end{cases} \tag{19}$$

Then, we find from the application of the definition 15 for the system (11), for the MSIR model and the following system is obtained, respectively for the given equations.

$$\begin{cases} s^\alpha \hat{M}(s) - s^{\alpha-1} M(0) &= \frac{\Lambda}{s} - (\delta + \mu) \hat{M}(s) \\ s^\alpha \hat{S}(s) - s^{\alpha-1} S(0) &= \delta \hat{M}(s) - \frac{\beta}{N} \mathcal{SI} - \mu \hat{S}(s) \\ s^\alpha \hat{I}(s) - s^{\alpha-1} I(0) &= \frac{\beta}{N} \mathcal{SI} - (\gamma + \mu) \hat{I}(s) \\ s^\alpha \hat{R}(s) - s^{\alpha-1} R(0) &= \gamma \hat{I}(s) - \mu \hat{R}(s) \end{cases} \tag{20}$$

Then, we find from the application of the definition 15 for the system (12), for the MSEIR model and the following system is obtained, respectively for the given equations.

$$\begin{cases} s^{\alpha}\hat{M}(s) - s^{\alpha-1}M(0) &= \frac{\Lambda}{s} - (\delta+\mu)\hat{M}(s) \\ s^{\alpha}\hat{S}(s) - s^{\alpha-1}S(0) &= \delta\hat{M}(s) - -\frac{\beta}{N}\mathcal{SI} - \mu\hat{S}(s) \\ s^{\alpha}\hat{E}(s) - s^{\alpha-1}E(0) &= \frac{\beta}{N}\mathcal{SI} - (\epsilon+\mu)\hat{E}(s) \\ s^{\alpha}\hat{I}(s) - s^{\alpha-1}I(0) &= \frac{\beta}{N}\mathcal{SI} - (\gamma+\mu)\hat{I}(s) \\ s^{\alpha}\hat{R}(s) - s^{\alpha-1}R(0) &= \gamma\hat{I}(s) - \mu\hat{R}(s) \end{cases} \tag{21}$$

Finally, we find from the application of the definition 15 for the system (13), for the double SI model, in the proposed case for Zika virus. For uniformity in writing complex equations after the Laplace transform, we will need the following integrals, with similar reasoning to the equation (17).

$$\mathcal{SI}_{hh} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{Re(z)-iT}^{Re(z)+iT} \hat{S}_h(z)\hat{I}_h(s-z)dz$$

$$\mathcal{SI}_{hm} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{Re(z)-iT}^{Re(z)+iT} \hat{S}_h(z)\hat{I_m}(s-z)dz$$

$$\mathcal{SI}_{mh} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{Re(z)-iT}^{Re(z)+iT} \hat{S_m}(z)\hat{I}_h(s-z)dz$$

So, through the Laplace transformation the following system is obtained, respectively for the given equations.

$$\begin{cases} s^{\alpha}\hat{S}_h(s) - s^{\alpha-1}S_h(0) = \frac{\Lambda_h}{s} - \beta_1\mathcal{SI}_{hh} - \beta_2\mathcal{SI}_{hm} - k_1\hat{S}_h(s) \\ s^{\alpha}\hat{I}_h(s) - s^{\alpha-1}I_h(0) = \beta_1\mathcal{SI}_{hh} + \beta_2\mathcal{SI}_{hm} - k_1\hat{I}_h(s) \\ s^{\alpha}\hat{S_m}(s) - s^{\alpha-1}S_m(0) = \frac{\Lambda_m}{s} - \mu\mathcal{SI}_{mh} - k_2\hat{S_m}(s) \\ s^{\alpha}\hat{I_m}(s) - s^{\alpha-1}I_m(0) = \mu\mathcal{SI}_{mh} - k_2\hat{I_m}(s) \end{cases} \tag{22}$$

In the above systems, all initial conditions are known, depending on the virus and the model we are studying each time, in terms of the data given.

3.3. Representation of the results of transformations in matrices

Expressing the equations of Laplace transforms in tabular form can offer many advantages and insights. By representing the equations in a matrix, we can utilize the powerful tools and techniques of linear algebra to analyze and solve the system more efficiently. A key advantage is the ability to apply matrix operations and manipulations to the equations. This allows us to perform operations such as matrix inversion, calculating determinants and eigenvalue analysis, which can help us understand the stability, controllability and observability of the system.In addition, the matrix representation allows us to use existing algorithms and numerical methods to solve systems of equations. Techniques such as Gaussian elimination, LU decomposition and eigenvalue decomposition can be easily applied to analyze system behavior and response.Another advantage is the concise and organized representation of the equations. By arranging the variables and coefficients in a table, we can clearly see the relationships and dependencies between the variables. This can help identify patterns, simplify equations and provide a more intuitive understanding of system dynamics. In addition, expressing the equations in tabular form facilitates the application of Laplace transformation techniques to study the frequency response of the system. The matrix representation allows us to manipulate and transform the equations directly using Laplace field operations such as multiplication, addition and differentiation. In summary, the expression of the Laplace transformation equations in matrix form offers computational efficiency, analytical tools and a structured representation of the system. It allows us to apply linear algebra techniques, exploit existing numerical methods and gain insights into the behaviour and properties of the system. This matrix-based approach enhances our ability to analyze, solve and understand complex systems described by Laplace transforms.

More specifically, the models we saw before will be expressed in the form of analytical matrices. As we have seen in the formula (15) the values $s, \alpha, n,$, so we will study only Laplacian and the complete result is obtained by simple substitution and application.

For the SIR model, from the system (16) we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 \\ 0 & s^{\alpha} & 0 \\ 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}S(0) \\ s^{\alpha-1}I(0) \\ s^{\alpha-1}R(0) \end{bmatrix} = \begin{bmatrix} -\mu & -\frac{\beta}{N}\mathcal{SI} & 0 \\ 0 & -(\gamma+\mu) & 0 \\ 0 & \gamma & -\mu \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} + \begin{bmatrix} \frac{\Lambda}{s} \\ 0 \\ 0 \end{bmatrix} \quad (23)$$

For the SEIR model, from the system (18), we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 & 0 \\ 0 & s^{\alpha} & 0 & 0 \\ 0 & 0 & s^{\alpha} & 0 \\ 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{E}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}S(0) \\ s^{\alpha-1}E(0) \\ s^{\alpha-1}I(0) \\ s^{\alpha-1}R(0) \end{bmatrix} = \begin{bmatrix} -\mu & \frac{\beta}{N}\mathcal{SI} & 0 & 0 \\ \frac{\beta}{N}\mathcal{SI} & -(\mu+\alpha) & 0 & 0 \\ 0 & \alpha & -(\gamma+\mu) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{E}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} + \begin{bmatrix} \frac{\mu N}{s} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (24)$$

For the SEI model, from the system (19), we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 \\ 0 & s^{\alpha} & 0 \\ 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}S(0) \\ s^{\alpha-1}E(0) \\ s^{\alpha-1}I(0) \end{bmatrix} = \begin{bmatrix} -\mu & 0 & -\frac{\beta}{N}\mathcal{SI} \\ 0 & -(\epsilon+\mu) & \epsilon \\ 0 & 0 & -(\gamma+\mu) \end{bmatrix} \begin{bmatrix} \hat{S}(s) \\ \hat{E}(s) \\ \hat{I}(s) \end{bmatrix} - \begin{bmatrix} \frac{\Lambda}{s} \\ 0 \\ 0 \end{bmatrix} \quad (25)$$

For the MSIR model, from the system (20), we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 & 0 \\ 0 & s^{\alpha} & 0 & 0 \\ 0 & 0 & s^{\alpha} & 0 \\ 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{M}(s) \\ \hat{S}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}S(0) \\ s^{\alpha-1}E(0) \\ s^{\alpha-1}I(0) \\ s^{\alpha-1}R(0) \end{bmatrix} = \begin{bmatrix} \delta+\mu & 0 & 0 & 0 \\ \mu & \delta & -\frac{\beta}{N}\mathcal{SI} & 0 \\ 0 & \frac{\beta}{N}\mathcal{SI} & -(\gamma+\mu) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \begin{bmatrix} \hat{M}(s) \\ \hat{S}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} + \begin{bmatrix} \frac{\Lambda}{s} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (26)$$

For the MSEIR model, from the system (21), we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 & 0 & 0 \\ 0 & s^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & s^{\alpha} & 0 & 0 \\ 0 & 0 & 0 & s^{\alpha} & 0 \\ 0 & 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{M}(s) \\ \hat{S}(s) \\ \hat{E}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}M(0) \\ s^{\alpha-1}S(0) \\ s^{\alpha-1}E(0) \\ s^{\alpha-1}I(0) \\ s^{\alpha-1}R(0) \end{bmatrix} = \begin{bmatrix} \delta+\mu & 0 & 0 & 0 \\ \delta & \mu & -\frac{\beta}{N}\mathcal{SI} & 0 \\ 0 & \epsilon+\mu & \frac{\beta}{N}\mathcal{SI} & 0 \\ 0 & 0 & \gamma+\mu & \frac{\beta}{N}\mathcal{SI} \\ 0 & -\mu & 0 & \gamma \end{bmatrix} \begin{bmatrix} \hat{M}(s) \\ \hat{S}(s) \\ \hat{E}(s) \\ \hat{I}(s) \\ \hat{R}(s) \end{bmatrix} + \begin{bmatrix} \frac{\Lambda}{s} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (27)$$

For the double SI model of Zika virus transmission, from the system (22), we have the following matrix.

$$\begin{bmatrix} s^{\alpha} & 0 & 0 & 0 \\ 0 & s^{\alpha} & 0 & 0 \\ 0 & 0 & s^{\alpha} & 0 \\ 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \begin{bmatrix} \hat{S}_h(s) \\ \hat{I}_h(s) \\ \hat{S}_m(s) \\ \hat{I}_m(s) \end{bmatrix} - \begin{bmatrix} s^{\alpha-1}S_h(0) \\ s^{\alpha-1}I_h(0) \\ s^{\alpha-1}S_m(0) \\ s^{\alpha-1}I_m(0) \end{bmatrix} =$$

$$\begin{bmatrix} -\beta_1\mathcal{SI}_{hh} - \beta_2\mathcal{SI}_{hm} - k_1 & 0 & 0 & 0 \\ 0 & \beta_1\mathcal{SI}_{hh} + \beta_2\mathcal{SI}_{hm} - k_1 & 0 & 0 \\ 0 & 0 & -\mu\mathcal{SI}_{mh} - k_2 & 0 \\ 0 & 0 & 0 & \mu\mathcal{SI}_{mh} - k_2 \end{bmatrix} \begin{bmatrix} \hat{S}_h(s) \\ \hat{I}_h(s) \\ \hat{S}_m(s) \\ \hat{I}_m(s) \end{bmatrix} + \begin{bmatrix} \frac{\Lambda_h}{s} \\ 0 \\ \frac{\Lambda_m}{s} \\ 0 \end{bmatrix} \tag{28}$$

4. DISCUSSION

Another key reason for transforming the systems into matrices is the possible methods of simplifying them, either by the method of arbitrary functions (setting a function of our own for the functions $S, I, R, M, S_h, S_m, I_h, I_m$ and the operations between them), or by considering for certain conditions any of the functions as fixed. Such methodologies and ideas may be useful in future research on Caputo fractional derivatives and matrix resolution after Laplace transforms.

5. CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this paper, we try to give a relatively complete view on fractional calculus and more specifically on Caputo fractional derivatives. We study some basic, internationally widespread in the field of research, epidemiological models of ordinary differential equations and perform transformations of the systems, first to Caputo derivatives and then Laplace transformations. Finally, we transform the resulting equations into matrices. The main reason we did the last transformation is because of the possibilities in the management of matrices and the algebraic methods that lead to their solution. Also, the reason why we dealt with the Laplace transformation is its simplicity in computation compared to other transforms. By solving the systems/matrices in a rigorously mathematical way, it is possible to see the behaviour of the ODEs without the need to find numerical solutions through approximations and to avoid the construction of computational models using software and code.

The research community has not extensively addressed the potential of Caputo derivatives and, more specifically, there is no literature on solving Caputo fractional derivatives after Laplace transformation with classical methods of matrix solving. The format change gives us additional possibilities in the use of the system functions, as mentioned in 3.3. In future research we are going to study further the behavior of the double SI model, its transformation in Caputo derivatives, the Laplace transformation and the attempt to solve the matrix with dimensions 4×4. Some of the methods of simplifying a complex matrix are to consider some functions as constants which can probably lead to direct results about the behaviour and equilibrium of the system in question.

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REFERENCES

1. Minardi F. *Fractional Calculus: Theory and Applications*. Mathematics, 2018. V. 6. Article No. 145. doi: 10.3390/math6090145
2. Miller K. S., Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: Wiley, 1993.
3. Abel N.H. Opløsning af et par opgaver ved hjelp af bestemte integraler. *Magazin for Naturvidenskaberne*, 1823. V. I. No. 2. P. 55–68. (In Danish)
4. Caputo M. Linear Models of Dissipation whose Q is almost Frequency Independent—II. *Geophysical Journal International*, 1967, V. 13. No. 5. P. 529–539.
5. Sene N. Numerical methods applied to a class of SEIR epidemic models described by the Caputo derivative. In:*Methods of Mathematical Modeling. Infectious Diseases*. Eds. H. Singh, H. M. Srivastava, D. Baleanu. Academic Press, 2022. P. 23—40. doi: 10.1016/B978-0-323-99888-8.00003-6
6. Caputo M., Fabrizio M. On the notion of fractional derivative and applications to the hysteresis phenomena. *Meccanica*, 2017. V. 52. P. 3043—3052.
7. Rezapour S., Mohammadi H., Jajarmi A. A New Mathematical Model for Zika Virus Transmission. *Advances in Difference Equations*, 2020. Article No. 589(2020). doi: 10.1186/s13662-020-03044-7
8. Debbouche N., Ouannas A., Batiha I. M., Grassi G. Chaotic dynamics in a novel COVID-19 pandemic model described by commensurate and incommensurate fractional-order derivatives. *Nonlinear Dynamics*, 2022. V. 109. P. 33—45.
9. Abbes A. Ouannas A., Shawagfeh N., Jahanshahi H. The fractional-order discrete COVID-19 pandemic model: stability and chaos. *Nonlinear Dynamics*, 2023. V. 111. P. 965–983.
10. Tunç O., Tunç C. Solution estimates to Caputo proportional fractional derivative delay integro-differential equations. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 2023. V. 117. Article No. 12. doi: 10.1007/s13398-022-01345-y
11. Hethcote H.W. Three Basic Epidemiological Models. In: *Applied Mathematical Ecology*. Ed. Levin S.A., Hallam T.G., Gross L.J. Berlin: Springer,1989. (Biomathematics Series, V. 18). doi: 10.1007/978-3-642-61317-3_5
12. Ross R. An application of the theory of probabilities to the study of a priori pathometry.—Part I. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*. 1916. V.92. No. 638. P. 204—230.
13. Ross R., Hudson H. An application of the theory of probabilities to the study of a priori pathometry.—Part II. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*. 1917. V.93. No. 650. P. 212—225.
14. Ross R., Hudson H. An application of the theory of probabilities to the study of a priori pathometry.—Part III. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*. 1917. V. 89. No. 621. P. 225—240.
15. Kermack W.O., McKendrick A.G. A Contribution to the Mathematical Theory of Epidemics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*. 1927. V.115. No. 772. P. 700—721.
16. Kendall D.G. Deterministic and stochastic epidemics in closed populations. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability: Contributions to Biology and Problems of Health*. 1956. V. 4. P. 149—165.
17. Engelmann L. A box, a trough and marbles: How the Reed-Frost epidemic theory shaped epidemiological reasoning in the 20th century. *History and Philosophy of the Life Sciences*. 2021. V. 43. Article No. 105. doi: 10.1007/s40656-021-00445-z

18. Baleanu D., Aydogn S.E., Mohammadi H., Rezapour S. On modelling of epidemic childhood diseases with the Caputo-Fabrizio derivative by using the Laplace Adomian decomposition method. *Alexandria Engineering Journal*. 2020. V. 59. No. 5. P. 3029–3039. doi: 10.1016/j.aej.2020.05.007
19. Aghdaoui H., Tilioua M., Nissar K. S., Khan I. A Fractional Epidemic Model with Mittag-Leffler Kernel for COVID-19 *Mathematical Biology and Bioinformatics*. 2021. V. 16. No. 1. P. 39-56. doi: 10.17537/2021.16.39
20. Dablander F. *Infectious diseases and nonlinear differential equations*. 2020. URL: https://fabiandablander.com/r/Nonlinear-Infection.html (accessed 22.03.2024).
21. Dobrushkin V., Gourley R. *The Laplace Transform*. In: Brown University Applied Mathematis. 2016. URL: https://www.cfm.brown.edu/people/dobrush/am33/MuPad/MuPad9.html (accessed 22.03.2024).
22. Jarad F., Abdeljawad T. Generalized fractional derivatives and Laplace transform. *Discrete and Continuous Dynamical Systems - S*. 2020. V. 13. No. 3. P. 709-722.
23. Samko S. G., Kilbas A. A., Marichev O. I. *Fractional Integrals and Derivatives: Theory and Applications*. Switzerland: Gordon and Breach,1993.
24. Luchko Y. Fractional Derivatives and the Fundamental Theorem of Fractional Calculus. *Fractional Calculus and Applied Analysis*. 2020. V. 23. P. 939–966.

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